10th Sozopol Workshop, Primorsko, Bulgaria, 5th June, 2018

Solar and Stellar Magnetohydrodynamics

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Part I

Basic Magnetohydrodynamics

Maxwell's Equations to the MHD Induction Equation

- Maxwell's Equations:

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}
$$

- The displacement current term drops out for non-relativistic plasma, and in systems with low-frequency (in Ampere's law)
- For a one fluid, charge-neutral model, Poisson's eqn. is redundant (net ρ non-existent)
- Ohm's law: $J = \sigma(E + v \times B)$

- Now you have all the ingredients for the induction equation (Derive!)

Describing a MHD System: Magnetic Field Induction Equation

- The Induction Equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

- The magnetic diffusivity η is also often expressed as $\lambda = 1/\mu\sigma$, where σ is the conductivity of the plasma

- And the field is divergence free (flux is conserved)

$$\nabla \cdot \mathbf{B} = 0$$

Describing a MHD System: Navier-Stokes Equations

- Navier-Stokes Equation

Fluid Counterpart

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(p \boxed{}\right) + \boxed{} + \mathbf{g} + \nu \nabla^2 \mathbf{v}$$

- Kinematic viscosity, $\upsilon = \mu / \rho$ (where μ = viscosity)

- Captures momentum conservation

- Rate of change of momentum = influx of momentum + pressure forces + surface forces + body forces + viscous forces

- Gravity is often neglected, as other forces dominate

- Where does the magnetic field terms come from?

Describing a MHD System: Energy Equation

- Energy Equation

$$\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla) p + \gamma p \nabla \cdot \mathbf{v} = Q$$

- Expresses conservation of energy and balance of heat fluxes
- Q encompasses the cumulative effects of heat gain and losses; in the MHD case, an Ohmic dissipation term is added, which is j^2 / σ
- Most often we simply ignore the energy equation in dealing with an MHD system: Through intelligent use of boundary conditions (governing flows), invoking the fact that energy is actually stored within magnetic fields themselves
- So dealing with B suffice when we have some information on the system…

Describing a MHD System: Continuity Equation

- Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Captures mass conservation:

$$\underbrace{\frac{\partial}{\partial t}\int_V \rho \, dv}_{\text{Rate of change of mass}} = \underbrace{-\oint_S \rho \boldsymbol{u} \cdot \boldsymbol{n} \, ds}_{\text{Net inflow of mass}}$$

- In plasma systems, it really is the summation of the charge (electron and ion density) conservation equations which have the same form

Assumptions for the MHD Model of Plasma

- Relativistic effects are not important

- Collision frequency small compared to plasma frequency

$$\omega_{pe}^2 = \frac{n_0 e^2}{m_e \varepsilon_0}$$

- Plasma is a continuum (system scale L >> ion gyro radius)
  (One does not see a single charge in motion, or its oscillations)

- Plasma is in thermodynamic equilibrium
  (timescale of interest >> collision timescale, L >> mean-free path)

- Plasma is a single fluid (L >> Debye length)
  (Explain Debye shielding…)

Concept of Debye's Length

- The distance within which the effect of a charge is felt (beyond which, the plasma appears neutral and a single fluid…)

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_b T}{e^2 n_0}}$$

- Potential falls off as 1/r in vacuum, but in a plasma due to the restoring force which tries to maintain charge neutrality, the potential falls faster:

$$\phi = \frac{A}{r} e^{-r/\lambda_D}$$

- Where *A* is total charge; Effectively, beyond λ_D the charge is not felt!

- Solar core Debye length ~ 10^{-11} m, solar wind ~ 10 m

The Complete Suite of MHD Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi\rho} + \mathbf{g} + \nu\nabla^2\mathbf{v}$$

$$\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p + \gamma p \nabla \cdot \mathbf{v} = Q$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

$$\nabla \cdot \mathbf{B} = 0$$

The Induction Equation: Magnetic Reynolds Number

- Governing equation in MHD:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

- Magnetic Reynolds Number:

$$R_m = \frac{VB / L}{\eta B / L^2} = \frac{VL}{\eta}$$

When Diffusion Dominates

• Induction equation reduces to:

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} \qquad \rightarrow \qquad \frac{\mathbf{B}}{\tau} = \frac{\eta}{L^2}\mathbf{B}$$

• Solution: $B = B_0 \exp(-t/\tau)$, with $\tau = L^2 / \eta$

• Fields will simple decay and get mixed on a timescale of τ

• Fields will diffuse to remove inhomogeneity in field lines, direction of diffusion depends on gradient of field

The Ideal MHD Limit: Flux Freezing

- The diffusivity is small or the Reynolds number is very, very high…
- Induction equation reduces to: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})$$
- Flux freezing theorem follows if the ideal MHD equation holds:

$$\frac{d\Phi_B}{dt} = 0$$

- Magnetic flux threading a surface S is given by: $\Phi_B \equiv \oint \mathbf{B} \cdot \hat{n}\, dS$
- The flux across any given surface, remains invariant with time in an ideal plasma flow (Note analogy with fluid vorticity)

Compression, Expansion and Advection of Fields

- An analogy with the continuity equation (rewritten as):

$$\frac{\partial \rho}{\partial t} = -\rho \boldsymbol{\nabla} \cdot \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla} \rho$$

- The first term on the R.H.S. signifies density changes due to converging or diverging flows and the second term due to advection (bodily carriage of varying density profile by the flow)

- Similarly the ideal induction equation yields:

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{B}\, \boldsymbol{\nabla} \cdot \mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{v}$$

- These encompass the effect of compressive-diverging flows, advection and inductive (such as shearing) flows

MHD Equilibrium: Plasma Beta Parameter

- One needs to consider the momentum conservation equation where the balance of forces is implicitly present

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi\rho} + \mathbf{g} + \nu\,\nabla^2\mathbf{v}$$

- If gravity is ignored, and we set the d/dt terms and the velocity terms to 0, implying a equilibrium configuration then the system reduces to

$$\boldsymbol{\nabla} p = \frac{1}{4\pi}(\boldsymbol{\nabla} \times \mathrm{B}) \times \mathrm{B}.$$

- This is a gas pressure balanced scenario

- The plasma β parameter is the ratio of the gas pressure to the magnetic pressure and governs this balance…

Gas (Plasma Fluid) Pressure Dominating (High β Plasma)

- If the gas pressure p dominates then this can hold magnetic flux tubes together against expansion forming coherent flux tubes

- The gas can then also push and distort the magnetic fields and induct fields

- Within the Sun's interior this is what happens; density is very high so gas pressure is high ($P = \rho RT$)

- Solar and stellar dynamo mechanism works in this regime, where the fluids govern the dynamics and can channelize their energy into magnetic fields

Magnetic Pressure Dominating (Low β Plasma): Solar Corona

- If the gas pressure is really low (solar outer atmosphere), then we can neglect the gas pressure altogether and the system reduces to

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}$$

- Solution 1: $\nabla \times \mathbf{B} = \mathbf{0}$

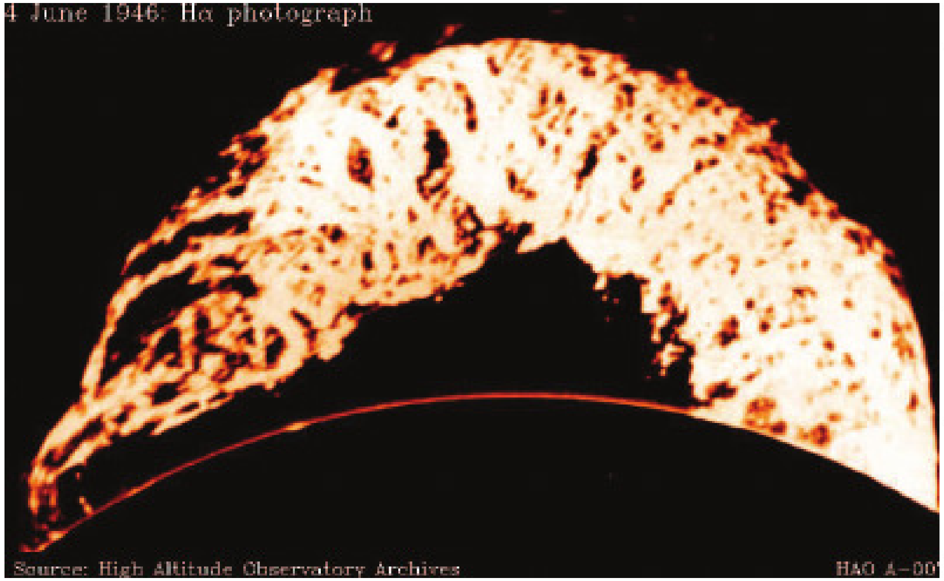
—implying no currents in the system (j = 0); potential field solution

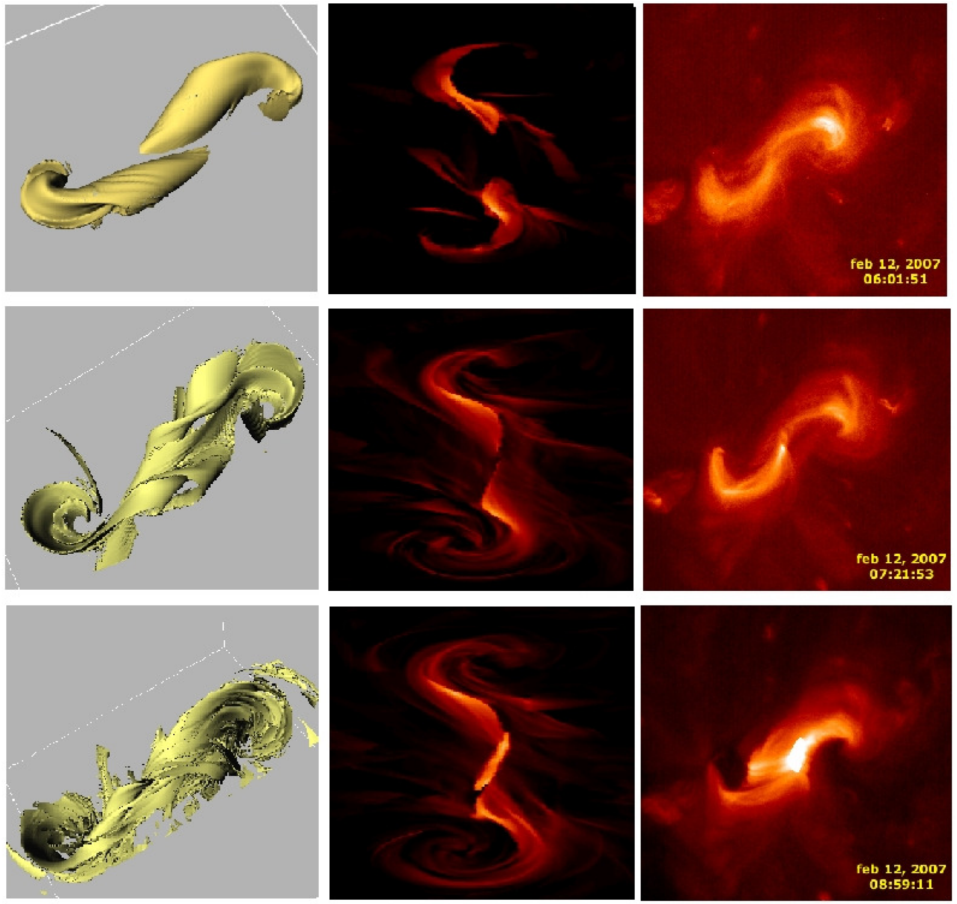
- Solution 2: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

—implying currents aligned in the direction of the field; the parameter alpha signifies twist in the field lines (check with a twisted flux tube)

- Magnetic fields lines can be twisted…! A component of helicity…

Examples of Solar Magnetic Field Twist

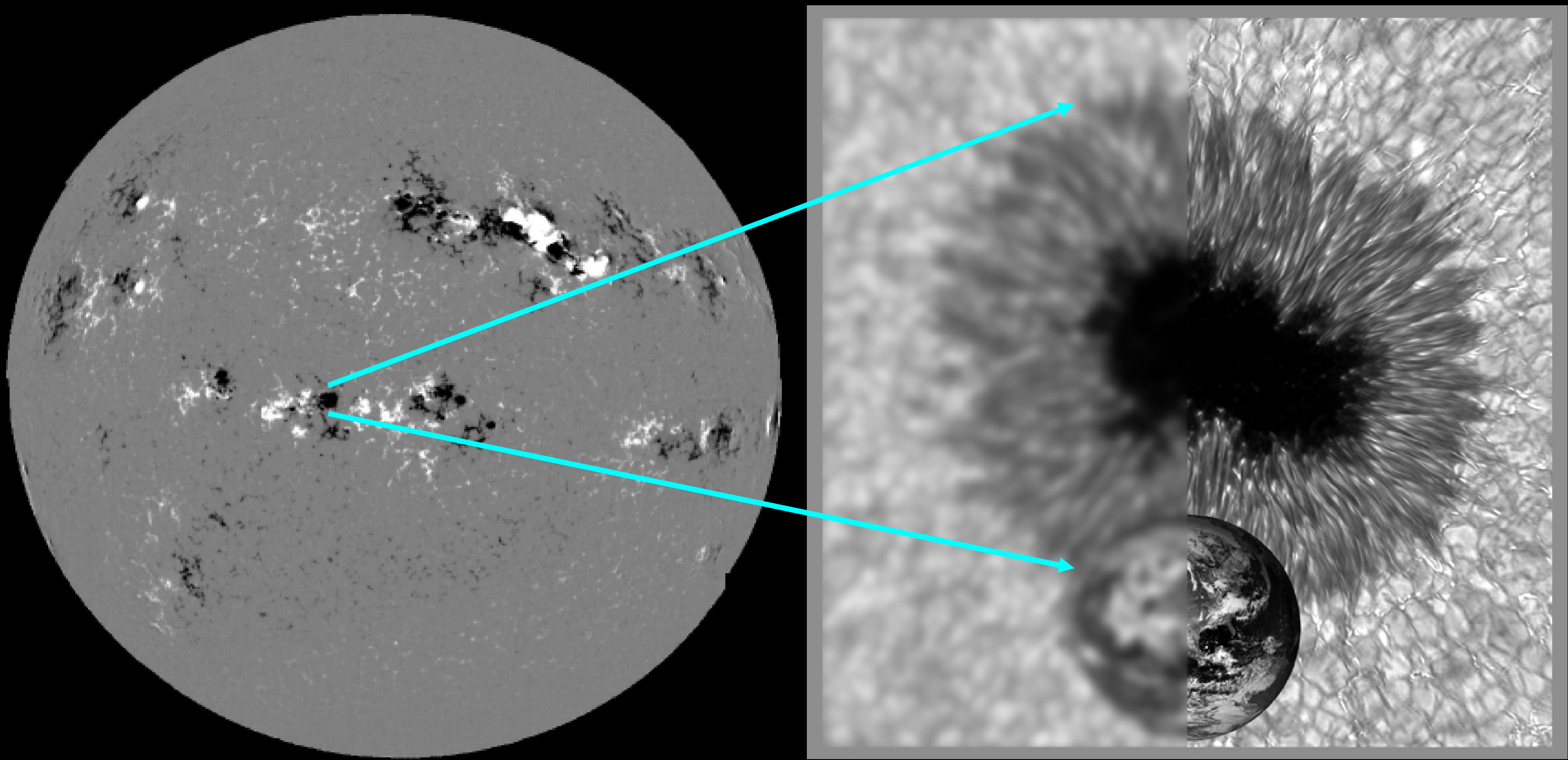




Part II

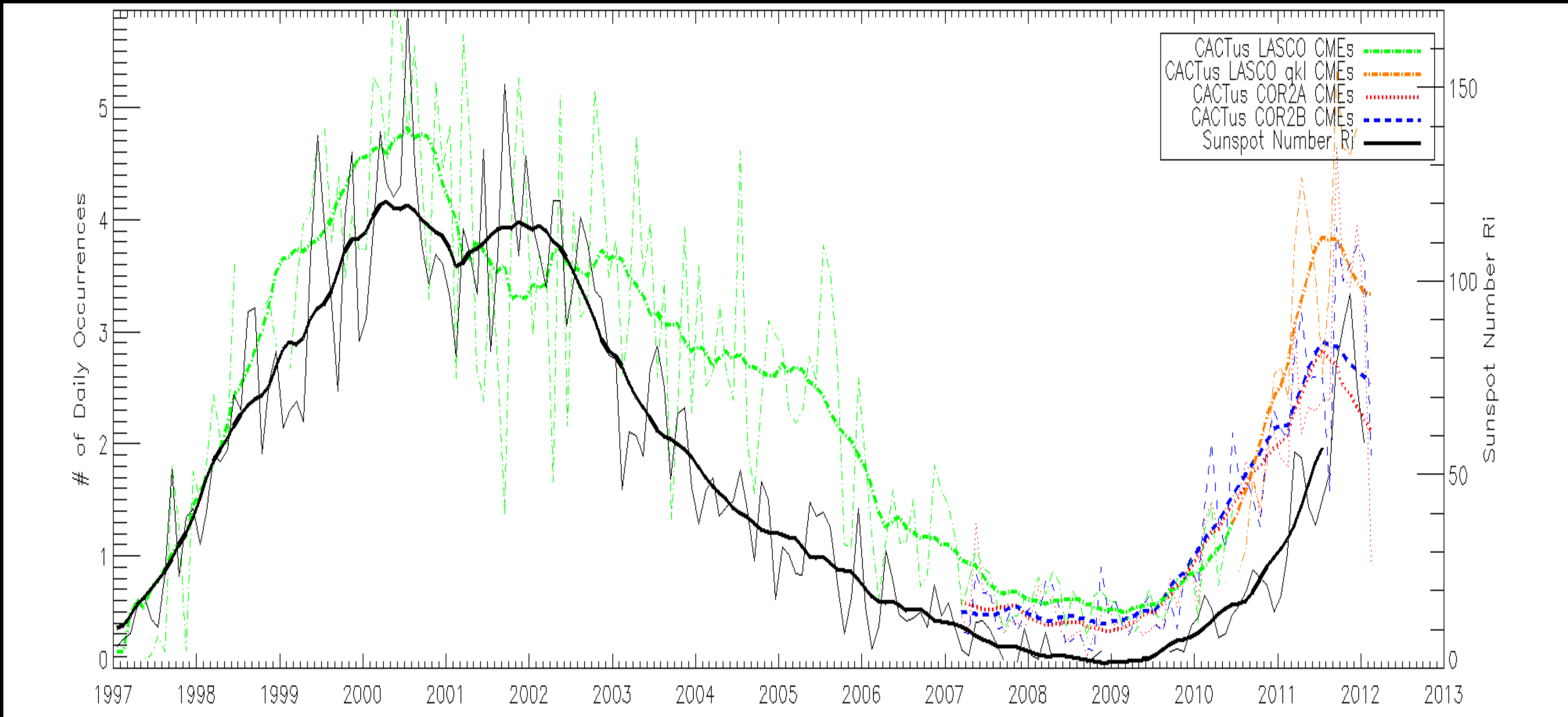
Solar Magnetic Cycle

Solar Magnetic Fields: Sunspots



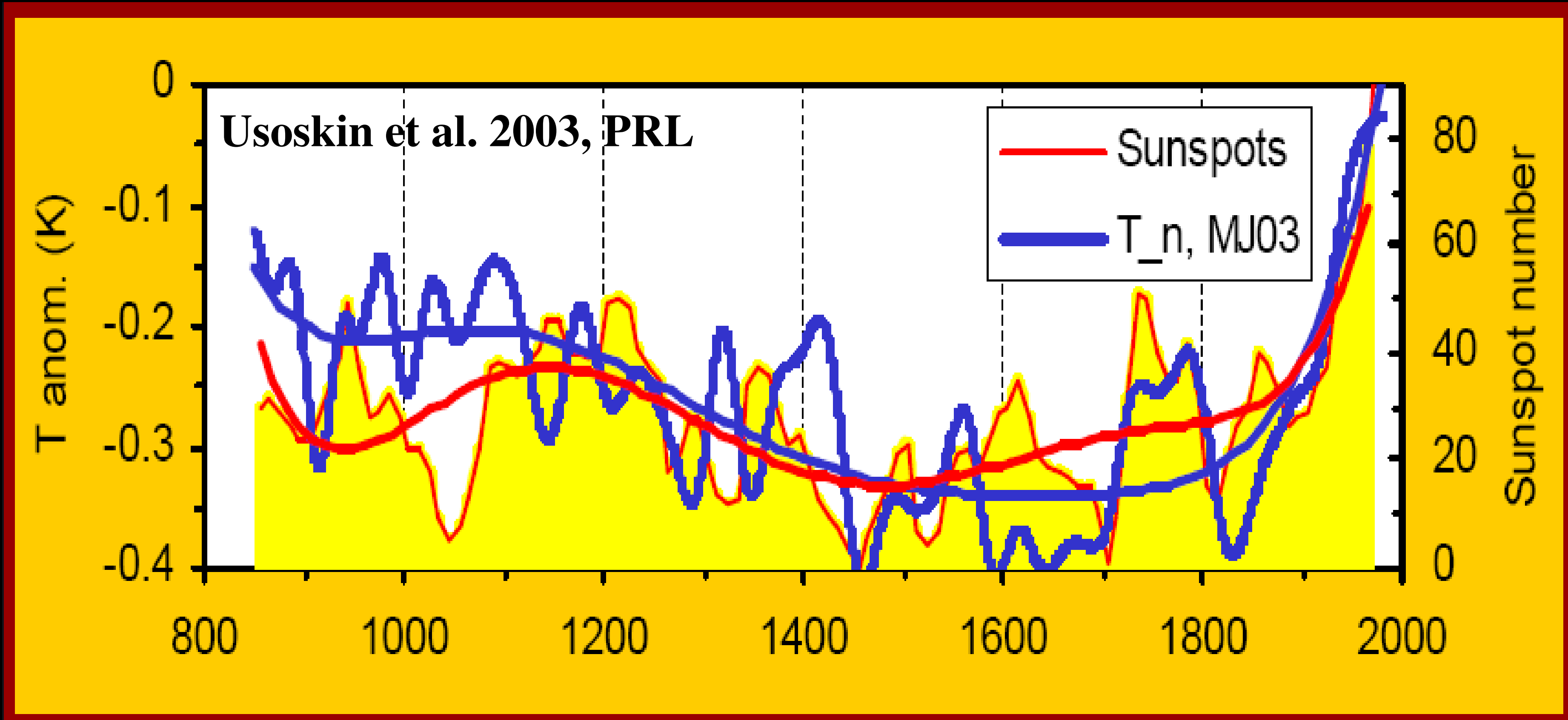
- First telescopic observations by Galileo and Scheiner (1611 AD)
- Size about 10,000 Km
- Sunspots are strongly magnetized ~ 1000 G (Hale 1908,ApJ)
- Appears dark because they inhibit convection

Sunspots are the Seats of Solar Storms



- Solar flares and coronal mass ejections (CMEs) – biggest explosions in the solar system – eject magnetized plasma and charged particles ($m \sim 10^{12}$ Kg, $v \sim 500$-2000 km/s, $E \sim 10^{24}$ Joules)
- Rate of solar storm occurrence correlated with sunspot cycle

The Cycle of Sunspots and its Relevance for Climate



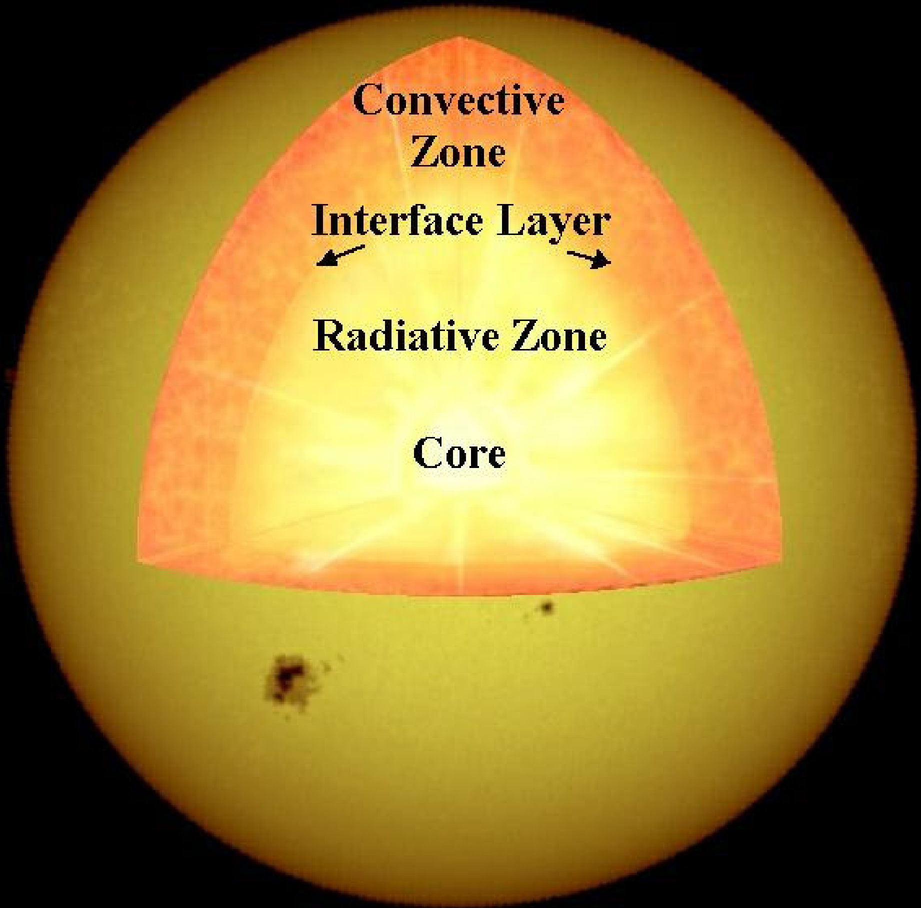
- Number of sunspots observed on the Sun varies cyclically
- Modulates the solar radiative energy output
- Primary natural energy input to the climate system
- Maunder minimum – the "little ice age" – suggestive of link

Understanding & Forecasting Solar Activity Important

Magnetic Fields
Solar Storms
Solar Wind Conditions
Solar Radiation Spectrum

Magnetic field output – the cycle of sunspots, govern other solar activity parameters

Window to the Solar Interior



- Matter exists in the ionized state in the solar interior
- Convection zone sustains plasma motion and magnetic fields
- Enter magnetohydrodynamics

Basic Physics: Plasma Flows Govern Magnetic Field Generation

- Governing equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

- Magnetic Reynolds Number:

$$R_m = \frac{VB/L}{\eta B/L^2} = \frac{VL}{\eta}$$

- In Astrophysical systems, R_M usually high, magnetic field creation possible and fields are frozen with the plasma
—Diffusion timescale τ_η > Flow timescale τ_v

- In solar interior, plasma $\beta >> 1$ (gas pressure higher than magnetic pressure and therefore, plasma flows govern field dynamics
— Solar Dynamo Models

Theoretical Concepts in MHD: The Anti-Dynamo Theorem

- Convective plasma motions may generate magnetic field by induction (Larmor 1919)

- The anti-dynamo theorem (Cowling 1934):

  An axisymmetric magnetic field cannot be
  maintained by a steady axisymmetric plasma flow

- Rules out dynamo action in systems with certain symmetries in the flow fields

- Can dynamos exist?

Theoretical Concepts in MHD: Around the Anti-Dynamo Problem

- Yes! There are ways around the anti-dynamo theorem

- First positive breakthrough came in 1955, when Eugene Parker postulated that helical turbulent motions can play a crucial role in generating astrophysical magnetic fields through the α-effect

- This is possible because the α-effect removes certain symmetry constraints on the flow fields

- The α-effect can work only in rotating systems, where Coriolis force exists and therefore results in non-zero helicity

- Fortunately, most astrophysical systems have rotation and, small-scale turbulent and large-scale structured motions

Theoretical Concepts in MHD: The Mean-Field Approximmation

- In a turbulent magnetized medium, the flow and field components can be expressed as a sum of fluctuating and mean components
- It turns out that the cross-product of the fluctuating components ($u' \times B'$) generates a mean electromotive force (which after some drastic truncations) can be expressed as:

$$\mathcal{E} = \alpha \langle \mathrm{B} \rangle - \eta_{\mathrm{T}} \nabla \times \langle \mathrm{B} \rangle$$

$$\alpha \sim -\frac{\tau_{\mathrm{c}}}{3} \langle \mathbf{u}' \cdot \nabla \times \mathbf{u}' \rangle$$

$$\eta_{\mathrm{T}} \sim \frac{\tau_{\mathrm{c}}}{3} \langle \mathbf{u}' \cdot \mathbf{u}' \rangle$$

- Where η is the turbulent diffusivity, α is the mean-field alpha-effect, u' is the fluctuating velocity and τ the correlation time of turbulence
- It is essentially this alpha effect term – related to the Coriolis force – that makes dynamo action possible by bypassing the anti-dynamo theorem

Building a Kinematic Solar Dynamo Model

- Axisymmetric Magnetic Fields:

$$\boldsymbol{B} = B\boldsymbol{e}_\phi + \nabla \times (A\boldsymbol{e}_\phi)$$

- Axisymmetric Velocity Fields:

$$\boldsymbol{v} = \boldsymbol{v}_p + r \sin\theta\Omega\boldsymbol{e}_\phi$$

- Plug these into the MHD induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta\nabla \times \mathbf{B}) + \nabla \times \mathcal{E}$$

And separate the two components to obtain…..

The Axisymmetric Dynamo Equations

- Magnetic field evolution

$$\frac{\partial\langle A\rangle}{\partial t} = \underbrace{(\eta+\eta_{\mathrm{T}})\left(\nabla^2 - \frac{1}{\varpi^2}\right)\langle A\rangle}_{\text{turbulent diffusion}} - \frac{\mathbf{u}_{\mathrm{p}}}{\varpi}\cdot\nabla(\varpi\langle A\rangle) + \underbrace{\alpha\langle B\rangle}_{\text{MFE source}},$$

$$\frac{\partial\langle B\rangle}{\partial t} = \underbrace{(\eta+\eta_{\mathrm{T}})\left(\nabla^2 - \frac{1}{\varpi^2}\right)\langle B\rangle}_{\text{turbulent diffusion}} + \underbrace{\frac{1}{\varpi}\frac{\partial\varpi\langle B\rangle}{\partial r}\frac{\partial(\eta+\eta_{\mathrm{T}})}{\partial r}}_{\text{turbulent diamagnetic transport}} - \varpi\mathbf{u}_{\mathrm{p}}\cdot\nabla\left(\frac{\langle B\rangle}{\varpi}\right) - \langle B\rangle\nabla\cdot\mathbf{u}_{\mathrm{p}}$$

$$+ \underbrace{\varpi(\nabla\times(\langle A\rangle\,\hat{\mathbf{e}}_{\phi}))\cdot\nabla\Omega}_{\text{shearing}} + \underbrace{\nabla\times[\alpha\nabla\times(\langle A\rangle\,\hat{\mathbf{e}}\,\phi)]}_{\text{MFE source}},$$

- Dynamo numbers (quantifies efficiency): $C_\alpha = \dfrac{\alpha_0 R_\odot}{\eta_0}, \qquad C_\Omega = \dfrac{(\Delta\Omega)_0 R_\odot^2}{\eta_0}$

- Various forms: α^2, $\alpha^2\Omega$ and $\alpha\Omega$ dynamo equations follow from above

- Discuss relation between Dynamo number and Rossby Number

The αΩ Dynamo Equations

- Toroidal field evolution:

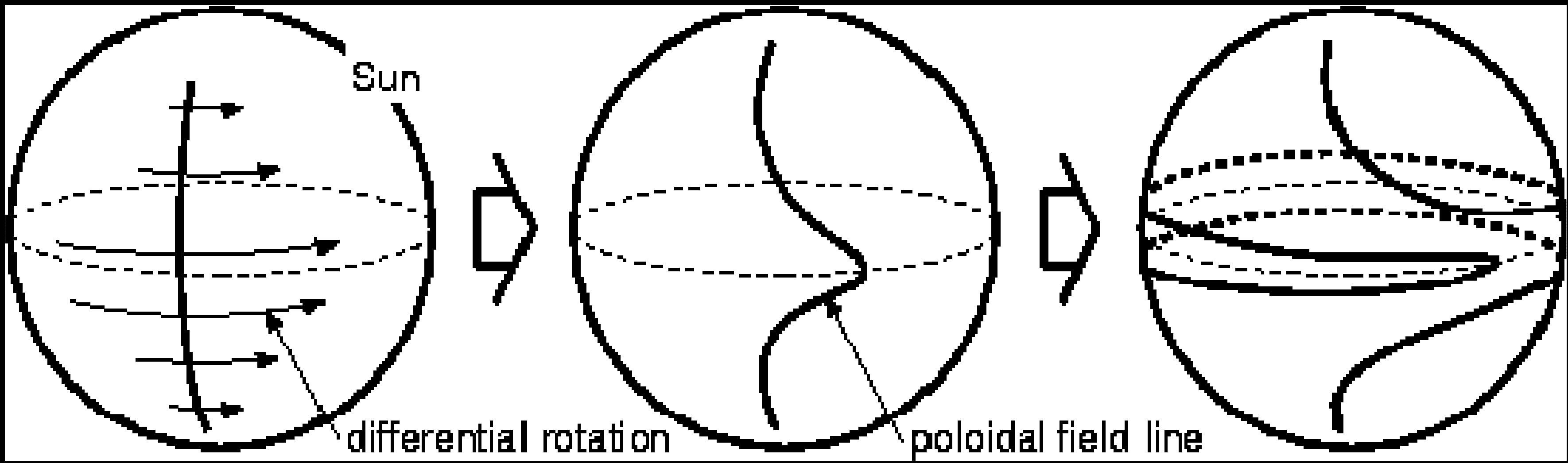
$$\frac{\partial B_\phi}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_r B_\phi\right)+\frac{\partial}{\partial \theta}\left(v_\theta B_\phi\right)\right]$$
$$=\eta\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)B_\phi+r\sin\theta\left(B_P.\nabla\right)\Omega-\nabla\eta\times\left(\nabla\times B_\phi\right)$$

- Poloidal field evolution:

$$\frac{\partial A}{\partial t}+\frac{1}{r\sin\theta}\left(v_P.\nabla\right)\left(r\sin\theta A\right)=\eta\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)A+S_\alpha$$

- Poloidal field source is parameterized by S_α
- Often, the alpha-term includes quenching, to limit field amplitude
- Buoyancy algorithm used to represent the emergence of ARs

Current Understanding: Toroidal Field Generation (Omega Effect)



Poloidal field

Toroidal Field

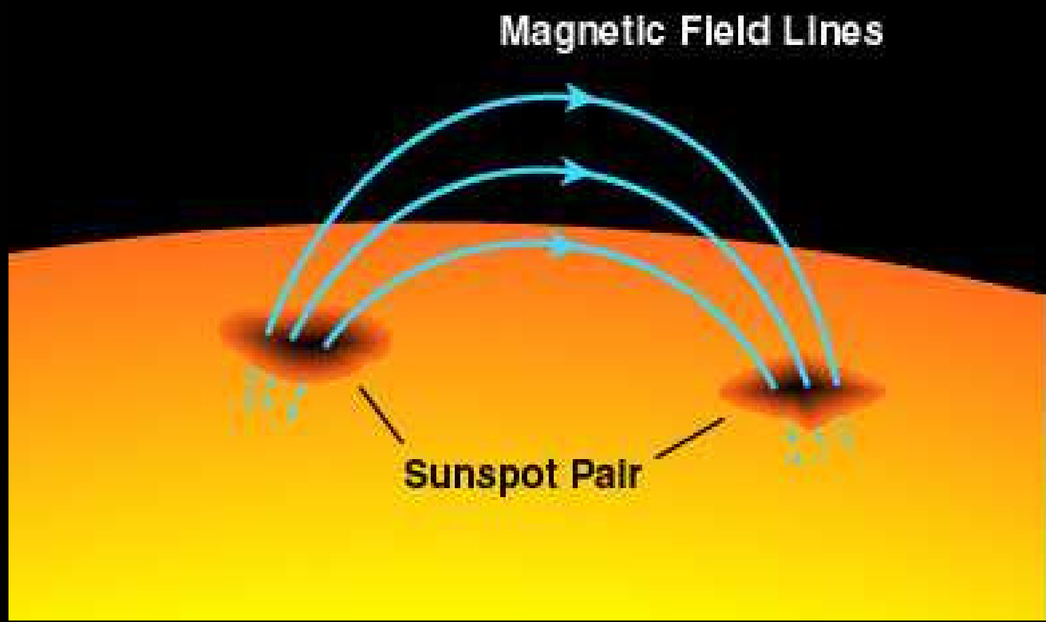
- Differential rotation will stretch a pre-existing poloidal field in the direction of rotation – creating a toroidal component (Parker 1955,ApJ)

Magnetic Buoyancy and Sunspot Formation

- Stability of Toroidal Flux Tubes – Magnetic Buoyancy (HW) (Parker 1955, ApJ)

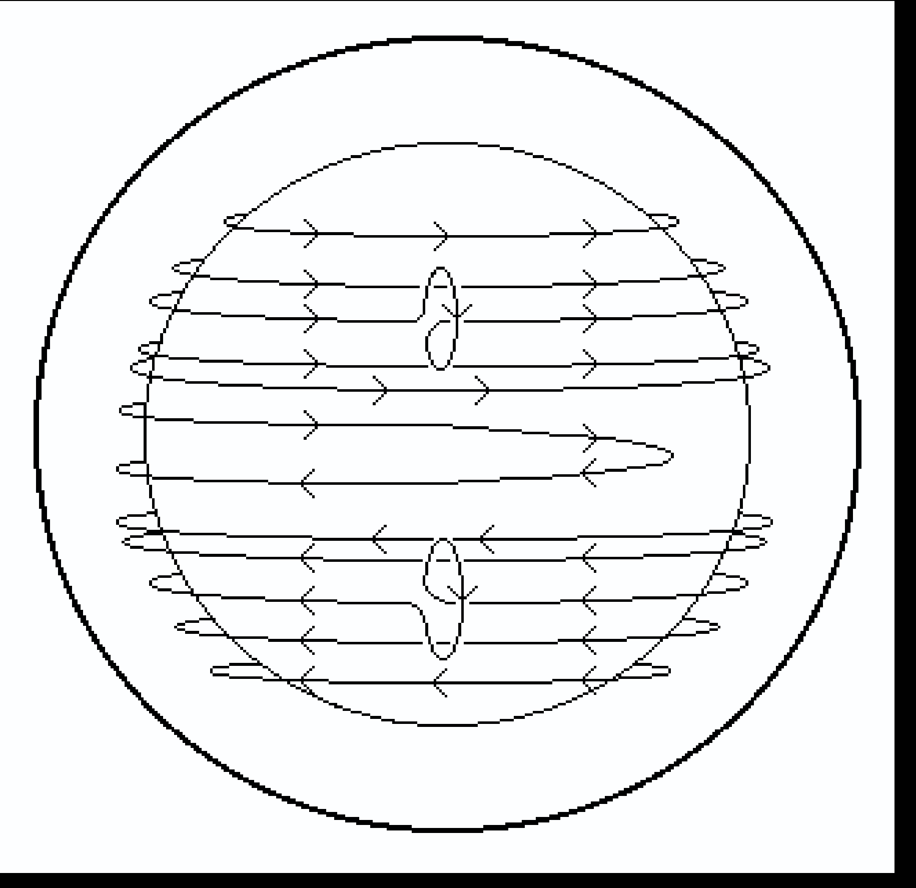
$$P_E = P_I + \frac{B^2}{8\pi}$$

$\rho_{\text{Internal}} < \rho_{\text{External}}$



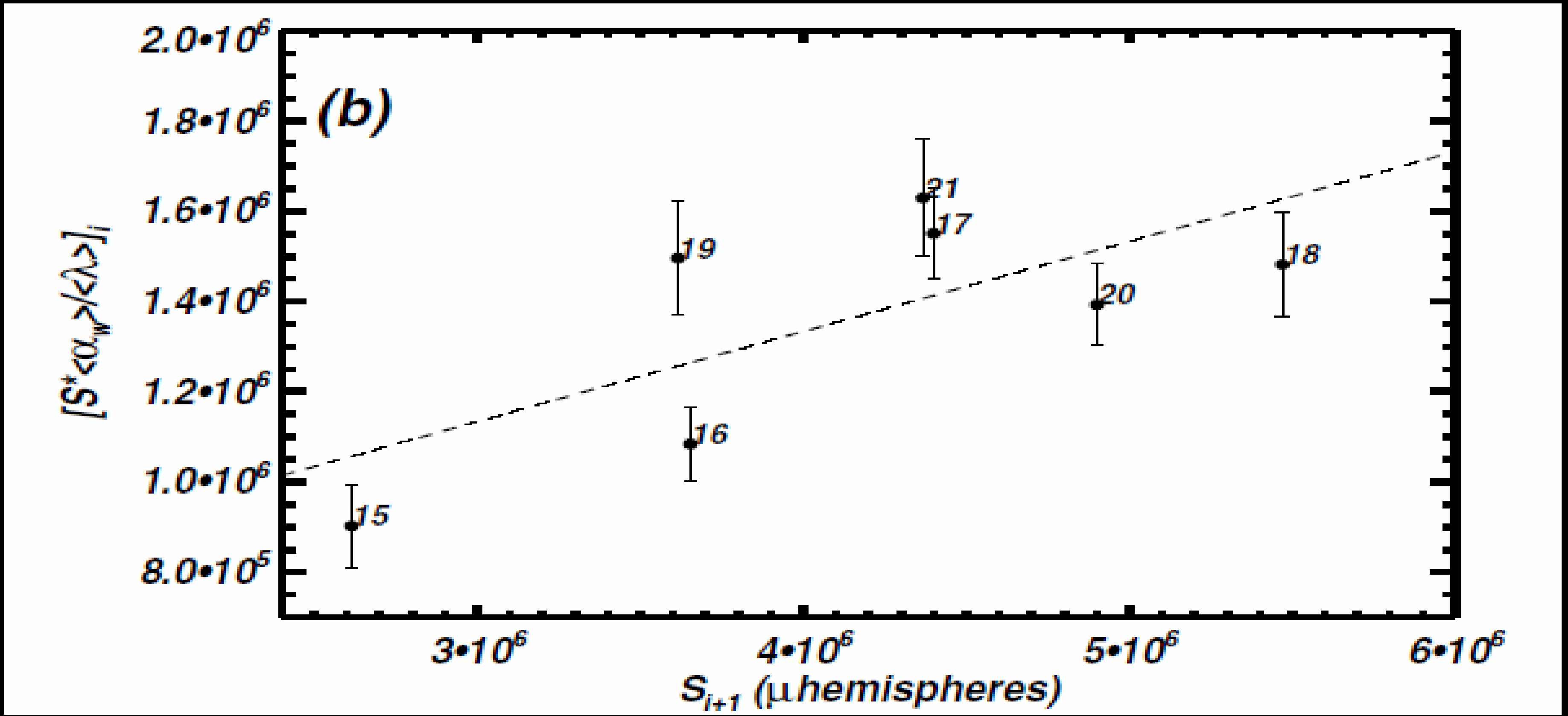
- Buoyant eruption, Coriolis force imparts tilts (sunspots are tilted)

Poloidal Field Generation – The MF α-effect



- Small scale helical convection – Mean-Field α-effect (Parker 1955)
- Buoyantly rising toroidal field is twisted by helical turbulent convection, creating loops in the poloidal plane
- Strong flux tubes will quench this mechanism, alternatives required…

Poloidal Field Generation: Tilted Bipolar Sunspot-Flux Dispersal



- Babcock (1961, ApJ) & Leighton (1969, ApJ) idea: tilted bipolar sunspots pairs decay and disperse near surface – <u>is observed</u>
- Numerous models have been constructed based on the BL idea — Strong observational support (Dasi-Espuig et al. 2010, A&A)

Summary

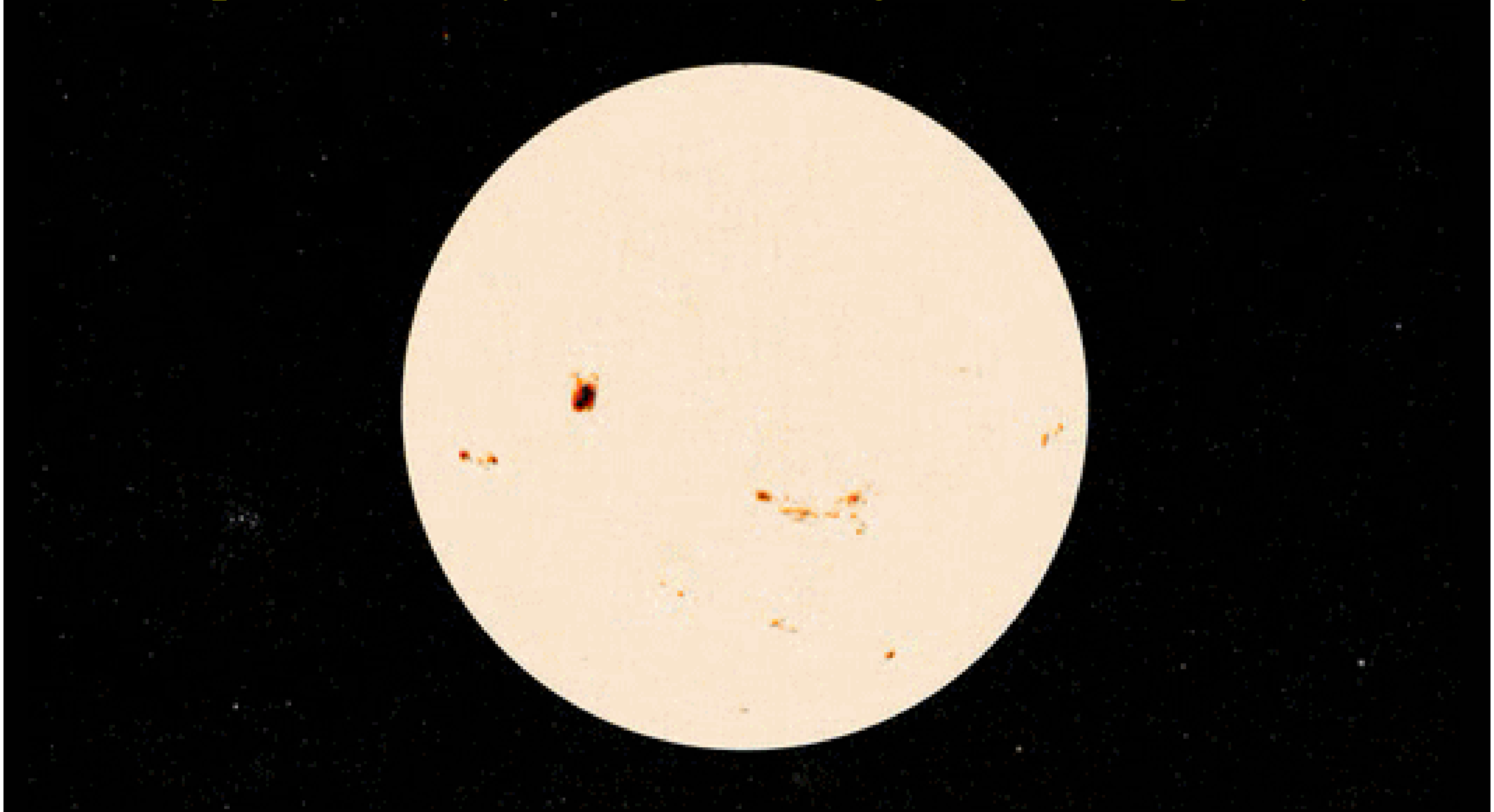
- MHD dynamo theory provides the theoretical basis for understanding the origin of astrophysical magnetic fields in plasma systems

- Understanding of the solar cycle is still incomplete; however, we are making progress…

- Predictive computational models are being developed which would have far reaching implications for assessing the space environment in the solar system

- The topics we talked about forms the basis of these developments

Computational Dynamo Modeling of the Sunspot Cycle



(Mũnoz-Jaramillo, Nandy & Martens, Nature, 2011)

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